TDP theory.

This is a translation done with a French/English translator. I apologize for my poor English.

Part 1

Introduction

As announced in my previous article on the forum, I present here the method of solving single-solution sudoku grids (puzzle) that I call "Technical of tracks", TDP in short (TDP = "Technique Des Pistes" in French).
This method is inspired by "Forcing chains" but differs from it by the fact that it treats the puzzle in a global way, and allows to develop a theory with its definitions, its properties and its theorems. It is a global approach to resolution in the same way as that of Allan Barker, or that of Denis Berthier.
From a practical point of view, it allows the sudokist to memorize only one way of doing things, which is simpler than knowing different advanced techniques that are not always easy to implement. In France, where I published a book and developed a website (http://www.assistant-sudoku.com), many sudokists use this method.
The purpose of this article, and the following ones, is to present the TDP and its arguments in as simple a way as possible. A complete document on this method can be found on my website, but alas in French.

Basic techniques

The TDP uses basic techniques (TB) for its implementation, so I specify here what I consider to be basic techniques. These are the ones that every sudokist knows in principle: unique candidates (singles), closed sets (pairs, triples, etc...) and alignments (pointing, box line).
We can eventually add X-wing for its simplicity, but nothing prevents those who want to add other techniques that they master well like fishs.

Let us give here a useful definition throughout the TDP.

K1, K2,... the cells that contain the candidates Ai of any set E={Ai, i=1,p} and by Ei the set of candidates of E contained in Ki. We have E=UEi. E'i refers to the complementary set of Ei in Ki.
The Ei sets are called the components of E. The E'i sets are called the counter-components of E.

Anti-track

Let's start with the notion of anti-track and its definition.

Definition :
- E = {Ai, i=1,2,...,n} being a set of n candidates Ai of the puzzle G, an anti-track P'(E) = {Bj, j=1,2,...,m} is the set of m candidates Bj that would be placed with the basic techniques (TB) in the cells of G as (true) solution IF the candidates of E were eliminated (false) from G. It is said that E generates P'(E).
- The pseudo-puzzle associated with P'(E) is the puzzle that would be obtained by placing the Bj candidates of P'(E).

Of course, by doing this, three situations can occur:
- or, the placement of candidates Bi encounters a contradiction (no candidate Bj possible in a box, two candidates Bj in the same area, etc...). It is then said that P'(E) is invalid.

- or, the candidates of P'(E) are all solution candidates. It is said that P'(E) is valid. We'll see later when this situation occurs.
- or, nothing can be concluded at this stage on the status of P'(E), because it cannot be proven that P'(E) is valid or invalid.

Construction Diagram :

The representation of an anti-piste is usually done by colour marking the candidates directly on the puzzle (see below), but it is also useful to explain the construction process by making a diagram connecting the candidates to each other as follows:

```
P'(E) : (-E)->B1->B2->[(B3->B4)->B5]->B6 …
                      \
                       ->B7->B8 …
```

which here means: the removal of candidates from E imposes the placement of B1, which imposes the placement of B2, which imposes the placement of B3, B4 and B5, which imposes the placement of B6, but also the placement of B2 imposes the placement of B7 and then B8, etc.

Here is a simple example of building an anti-track using on the puzzle a marking of the candidates by a yellow color.
E={3r2c1, 3r2c2}, I also write E=3r2c12.
P'(E)={9r2c2, 7r2c1, 9r3c5, 8r6c5, ...} whose construction diagram is:

```
P'(E)  : -(E)->9r2c2->9r3c5->8r6c5
            \          \
             ----------->7r2c1
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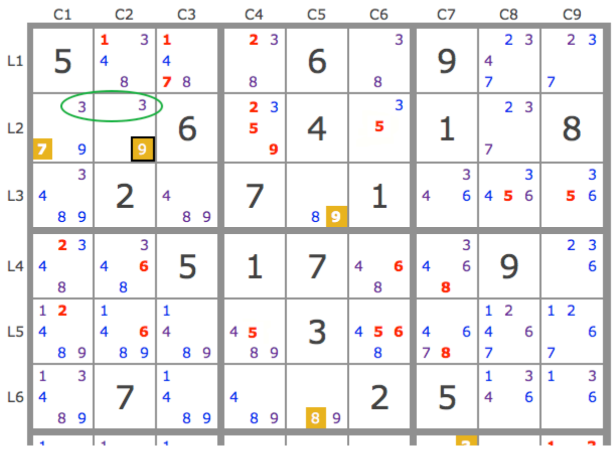

A first obvious and useful property that results from the very definition of an anti-runway is the following:

Theorem 1 :
If P'(E) is invalid, then at least one candidate Ai of E is a solution in his cell.
Consequently, any candidate C who sees all candidates Ai of E can be eliminated.*

** Two candidates A and C are said to see each other when they are either in the same cell or have the same occurrence and are in the same area (row, column or block).*

Indeed, if no candidate of E was a solution, then by definition all candidates of P'(E) would be solutions and P'(E) would be valid.

Let's enunciate a second theorem very useful for solving puzzles.

Theorem 2 :
If B is a candidate contained in P'(E), then any candidate C who sees B and all candidates from E can be eliminated.*

Indeed, if C were a solution in his cell, none of E's candidates would be a solution, therefore P'(E) would be valid and B would be a solution in his cell which contradicts the fact that B and C see each other. So C is not a solution in his cell and can be eliminated.

Here are some examples of using this theorem with the puzzles below where the candidates of the anti-track are marked in green.

<u>Example 1.</u>

E = {5r3c2, 2r1c1}. P'(E) = {5r1c2, 3r1c1, 2r1c8, 2r3c5, ... } whose construction diagram is:

```
               ->3r1c1->2r1c8->2r3c5
             /          /
P'(E) : -E->5r1c2--
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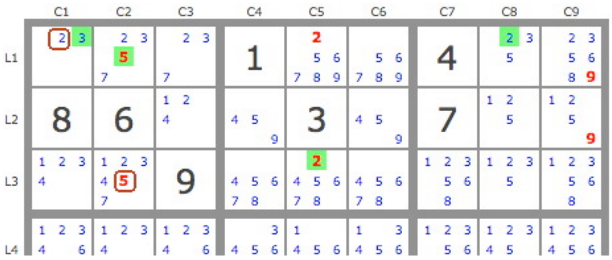

C=2r3c2 can be eliminated because it sees both candidates of E and B=2r3c5.

<u>Example 2.</u>

E = {2r3c2, 2r3c3}. P'(E) = {2r3c7, 5r2c7, 2r2c5, 2r1c12, ... } whose construction diagram is:

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P'(E) : -E->2r3c7->5r2c7->2r2c5->2r1c12
          \                              /
           -------------------------
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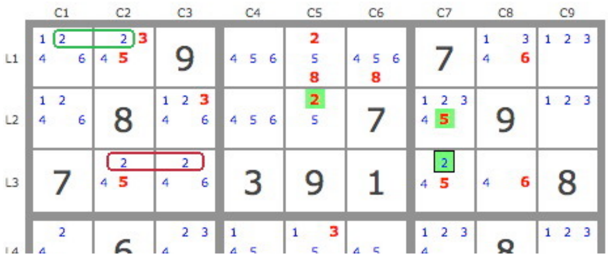
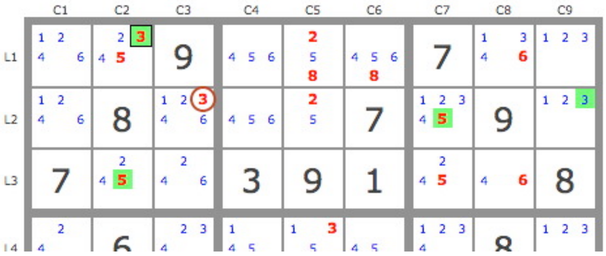


2r2c1 et 2r2c3 can be eliminated because it sees both candidates of E and 2r1c1 ou 2r1c2 .

<u>Example 3.</u>

E = {3r2c3}. P'(3r2c3) = {3r1c2, 5r3c2, 5r2c7, 3r2c9, ...}, then 3r2c7 can be eliminated.



A result of the same type can be established using closed sets.
An anti-track can contain subsets of candidates that form closed sets as defined below :
A set of candidates E1={B1j, j=1.2,...} is a closed set (locked set) contained in P'(E) when E1 is a closed set of the pseudo-puzzle associated with P'(E).

Let us then demonstrate the following theorem.

Theorem 3 :
In the same zone Z (block, row or column), E1 and E2 are two distinct sets with two components each consisting only of candidates of occurrence a and b.
If any of the conditions H1, H2 or H3 below are met, then all candidates of occurrence a or b that are not contained in E1UE2 can be eliminated from zone Z.
- (H1) : P'(E1) and P'(E2) are invalid.
- (H2) : P'(E2) is invalid and E2 is a closed set contained in P'(E1).
- (H3) : E2 is a closed set contained in P'(E1) and E1 is a closed set contained in P'(E2).

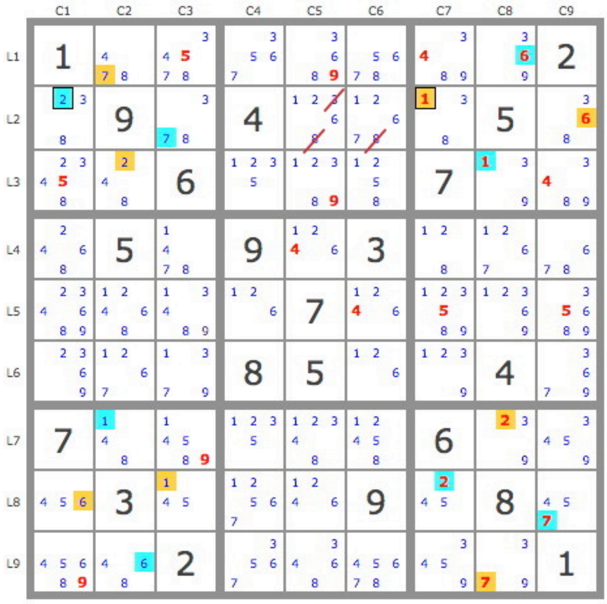
Demonstration :

1)- According to H1 and the theorem 1, E1 and E2 each contain a solution candidate, so the two candidates of Z of occurrence a and b are in E1UE2 => elimination of candidates of Z of occurrence a or b who are not in E1UE2 .

2)- According to H2 and the theorem 1, E2 contains at least one candidate solution and two cases are possible:
- P'(E1) valid => E2 contains two solution candidates, one of which is of occurrence a and the other b => elimination of candidates from Z of occurrence a or b who are not in E1UE2.
- P'(E1) invalid and this returns to case 1) => elimination of candidates from Z of occurrence a or b who are not in E1UE2.
3- According to H3, four cases are possible:
- P'(E1) valid => E2 contains two solution candidates, one of which is occurrence to the other of occurrence b => elimination of candidates from Z of occurrence a or b who are not in E1UE2.
- P'(E1) invalidates which leads to case 2) => elimination of candidates from Z of occurrence a or b who are not in E1UE2.
- Same reasoning and conclusion with P'(E2) depending on whether it is valid or invalid.
In all possible cases of H1, H2 and H3 candidates from Z of occurrence a or b that are not in E1UE2 are eliminated.

Here is an example on the famous Easter Monster puzzle of an application of all this.
G = 1.......2.9.4...5...6...7...5.9.3.......7.......85..4.7.....6...3...9.8...2.....1
Let's take E1={38r2c1, 38r2c3} and E2={38r2c7, 38r2c9}.

P'(E1) ={2r2c1, 7r2c1, 1r7c2, 6r9c2, 2r8c7, 7r8c9, 1r3c8, 6r1c8, ...} =>E2 is a closed set contained in P'(E1).
P'(E2) ={1r2c7, 6r2c9, 2r7c8, 7r9c8, 1r8c3, 6r8c1, 7r1c2, 2r3c2 ...} => E1 is a closed set contained in P'(E2).
According to the theorem3, 38r2c5 and 8r2c6 can be eliminated.



It is understood that the theorem also applies following the sets of type E1 and E2 which constitute the well-known Sk-loop of this puzzle and all eliminations are deduced from it.

But this theorem also applies to puzzles that do not count SK-loop, for partial eliminations.

The notion of anti-tracking is the basis of the TDP theory. We will see about this in the following.

Part2

Introduction

In this second part, I discuss the notion of conjugated tracks and their properties. Some of the terms used in this Part 2 have been defined in Part 1 to which I refer the reader.

Track and anti-track

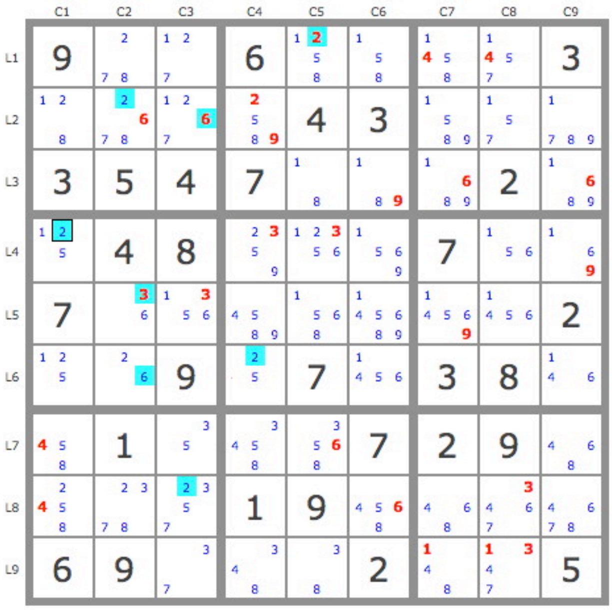
Part 1 has been used to define an anti-track and the counter-components of a set E. Then let's define a track P(E).

Let's start with a single Ak candidate, and designate by A'k all of Ak's complementary candidates in Ak's cell.

Definition:
The track P(Ak) is the set of candidates of the anti-track P'(A'k), so P(Ak)=P'(A'k).
It is the same to say that:
P(Ak)={Bj, j=1.2, ...} is the set of candidates Bj who would be placed with the basic techniques (TB) in the cells of G as (true) solution SI Ak was placed (true) in its cell.
It is said that Ak generates P(Ak).

Here is an example of how to build a P(Ak) track.
Ak=2r4c1
P(2r4c1)={2r1c4, 6r6c2, 3r5c2, 6r2c3,2r6c4, 2r1c5, 2r2c2, 2r8c3 ...}

What we directly represent the puzzle by a color marking, as below.



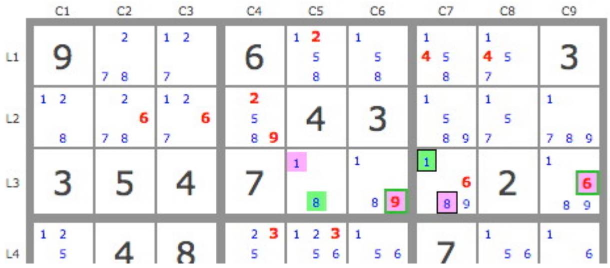
Then let's define a track P(E).

Definition :
E ={Ak, k=1,2,...n} being any set of candidates Ai of the puzzle G, a track P(E)={Bj, j=1,2, ...} is the set of candidates Bj common to the tracks P(Ak), i. e. obtained by intersection of the tracks P(Ak), so P(E)=∩P(Ak).
On dit que E génère P(E).

As with anti-track, several situations can occur during the construction of P(E) :
- either, the placement of Bj candidates encounters a contradiction (no possible Bj candidate in a cell, two Bj candidates in the same zone, etc.). It is then said that P(E) is invalid.
- or, the P(E) candidates are all solution candidates. It is said that P(E) is valid. We will look further when this situation occurs.
- or, nothing can be concluded at this stage about the status of P(E), because we do not know how to prove that P(E) is valid or invalid.

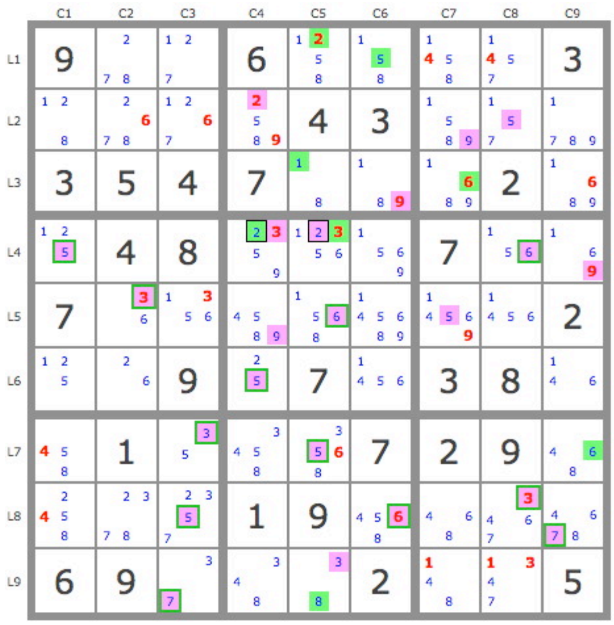
Let us give two examples of construction of P(E).

A first simple example with on the same puzzle E = {1r3c7, 8r3c7}, also noted 18r3c7.
P(E)=P(1r3c7)∩P(8r3c7) = {9r3c6, 6r3c9,…} which is made up of purple candidates surrounded by green common to both tracks P(1r3c7) marked in green and P(8r3c7) marked in purple. See figure below.



A second, more complex example.

On the same puzzle, E={2r4c4, 2r4c5} also noted E=2r4c45.
P(E)=P(2r4c4)P(2r4c5)={5r6c4, ...} which is made up of purple candidates surrounded by green common to both tracks P(2r4c4) marked in green and P(2r4c5) marked in purple. See figure below.



We can already state a property that is easy to demonstrate.

Theorem 1:
If P(E) is invalid, then none of the candidates Ai of E is a solution in his cell, so all candidates of E can be eliminated.

Indeed:
Invalid P(E) means that the placement of the Bj candidates common to the tracks P(Ak) whose P(E) is the intersection meets a contradiction, so each track P(Ak) meets this same contradiction too, i.e. is invalid. As P(Ak)=P'(A'k) where A'k is Ak's complementary set of candidates in his cell, the anti-tracks P'(A'k) are all invalid => each A'k contains a solution candidate (theorem 1 part 1) => each Ak can be eliminated .

Finally, for a set E of candidates from G, there are two types of candidate sets generated by E, a track P(E) and an anti-track P'(E) that satisfy the following property.

Property :
P(E) and P'(E) cannot be invalidated simultaneously.
If P(E) is invalid => P'(E) is valid, and if P'(E) is invalid => P(E) is valid.

This is because :
if P(E) and P'(E) were simultaneously invalid, then no candidate of E would be solution (Th 1 above) and at least one candidate of E would be solution (Th 1 part1), which is absurd.

Conjugated tracks

Language convention:
It will then be appropriate to write "Track P" to designate a track or an anti-track when it is not necessary to specify whether it is a track or an anti-track.

The track P(E) and the anti-track P'(E) are part of a set of pairs of "Tracks P" which are referred to as "onjugated tracks" and whose definition is as follows, in which the concepts of invalidity and validity are those previously given for the tracks and anti-tracks:

Definition :
A P1 Track and a P2 Track are conjugated when they cannot be invalided simultaneously. If Track P1 is invalid, then Track P2 is valid, and vice versa.

Thus, P(E) and P'(E) are conjugated according to the previous property, but there are other pairs of conjugated P tracks as shown in the following theorem :

Theorem 2 :
Let E1 and E2 be two distinct sets ($E1 \cap E2 = \emptyset$) of candidates from G.
If P'(E1UE2) is invalid, then P(E1) and P(E2) are conjugated.

Indeed:
if P(E1) and P(E2) were invalided simultaneously, then no candidate of E1 and E2, therefore of E1UE2, would be solution, which is absurd since P'(E1UE2) disabled implies that a candidate of E1UE2 at least is solution.

For these pairs of conjugated tracks that generalize the couple P(E)/P'(E) we can state the following two theorems.

Theorem 3 :
P1/P2 being a pair of conjugated tracks, any candidate from G who sees both a candidate from P1 Track and a candidate from P2 Track can be eliminated.*

** Two candidates A and B are said to see each other when they are either in the same cell or have the same occurrence and are in the same area (row, column or block).*

Indeed:
Let M be a candidate who sees both a candidate A from Track P1 and a candidate B from Track P2. If M is solution in his box, then A and B are not solutions => Track P1 and Track P2 are invalid, which is impossible. So M cannot be a solution and can therefore be eliminated.

This theorem has an obvious corollary, very useful in practice, which is the following:

Theorem 4:
P1/P2 being a couple of conjugated tracks, any candidate of G common to P1 Track and P2 Track is a solution in his cell.

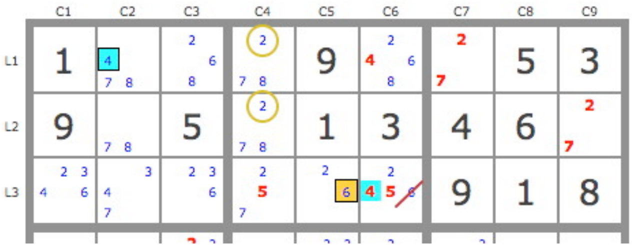
Indeed:
In the box of this candidate A common to P1 Track and P2 Track , all the other candidates in the box see A and can therefore be eliminated.

Here is a very simple example to illustrate these results.

G=....9..5.9.5..346.......9.871.6...4...4.5.6...6...7.218.1.......574..1...2..7....

On the puzzle simplified by the basic techniques (TB), we choose :

E1={4r1c2} and E2={2r12c4}, so E1UE2={4r1c2, 2r12c4}
Obviously, we have P'(E1UE2) invalid (forbidden rectangle 78r12c24) => P(E1) and P(E2) are conjugated tracks.
P(E1) ={4r1c2, 4r3c4, ...}.
P(E2)=P(2r1c4) $\cap$ P(2r2c4) = {6r3c6, ...}.
6r3c6 can be eliminated because it sees 6r3c5 and 4r3c6 (theorem 2).



The combined tracks are therefore the main tool for resolving the TDP since they allow for the validation and elimination of candidates.

Part3

Introduction
In this 3rd part, I present the notion of "opposite tracks" and the properties that go with it. I refer the reader to Parts 1 and 2, which define the concepts of runway and antirunway, validity and invalidity.

It should be remembered that by convention the term "Track P" refers to a track or an anti-track.

Opposite tracks

Definition :
A Track P1 and a Track P2 are opposite when there is at least one candidate from P1 who sees a candidate from P2 and vice versa.

A first property on the opposite tracks is as follows:

Theorem 1:
Two opposing tracks cannot be simultaneously valid.
If one is valid the other is invalid.

Indeed, if track P1 and track P2 were valid, the candidate from P1 and the candidate from P2 who see each other would both be solutions, which is impossible (in a single solution puzzle).

Theorem 2 :
If a Track Q1 is opposed to a Track P1, any conjugated Track P2 of Track P1 is included in Track Q1 (Track P 2 $\subseteq$ Track Q1).

To demonstrate this theorem, let us first demonstrate the following property:

If , track P1 valid => track P2 valid, then track P2 ⊆ track P1 .
Indeed,
This is due to the mode of the track construction mode.
Assuming or stating that a track is valid is like placing the candidates of the track on the puzzle by the basic techniques (TB) as if they were all solution candidates.
If placing those from Track P1 also allows us to say that we can place those from Track P2, then Track P1 is also built with the candidates from Track P2. So Track P2 ⊆ Track P1.

Therefore, the proof of the theorem is as follows:
Track P1 and Track P2 are conjugated, assuming that Track Q1 is valid => the opposite Track P1 is invalid => Track P2 is valid => Track P2 ⊆ Track Q1.

Le théorème suivant est un corollaire du théorème 2.

Théorème 3 :
Si deux pistes conjuguées Q1 et Q2 sont opposées à la même piste P1, toute piste conjuguée P2 de la piste P1 est valide.

Indeed,
If Track Q1 and Track Q2 are opposite to Track P1 => Track P2⊆Track Q1 and Track P2 ⊆Track Q2, since Track P2 is combined with Track P1. As Track Q1 or Track Q2 is valid => Track P2 is valid as being made up of all solution candidates.

The opposite tracks are an effective tool for solving difficult puzzles by using theorem 2, more rarely theorem 3.

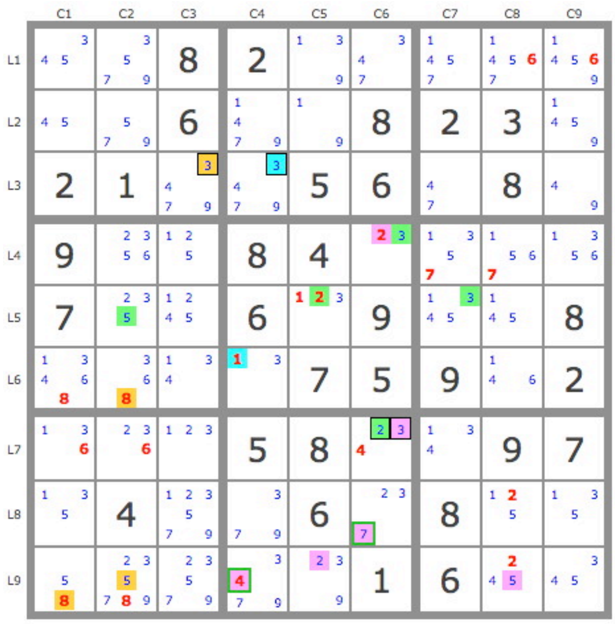
To illustrate this part here is an example of a resolution with the puzzle .
G = ..82.......6....3.21..56.8.9..84....7..6.9..8....75..2...58..97.4....8.......16..

After reducing the puzzle with the basic techniques (TB), I choose the pair 3r3 to build two conjugated tracks.
Tracks P(3r3c3) and P(3r3c4) are conjugated because P'(3r3c34) is obviously invalid.
P(3r3c3) ={3r3c3, 8r6c2, 5r9c2, 8r9c1,...}.
P(3r3c4) ={3r3c4, 1r6c4,...} contains the set 3r6c123.
As a result we can eliminate 3r45c3 which sees 3r3c3 and 3r6c123.
This is equivalent to a Finned X-Wing on all 3.

We can also eliminate 3r9c1 because P(3r9c1) is invalid in block b7. This is equivalent to an ALS-XZ.
This is to show you that we can do with TDP, the same as with advanced techniques.

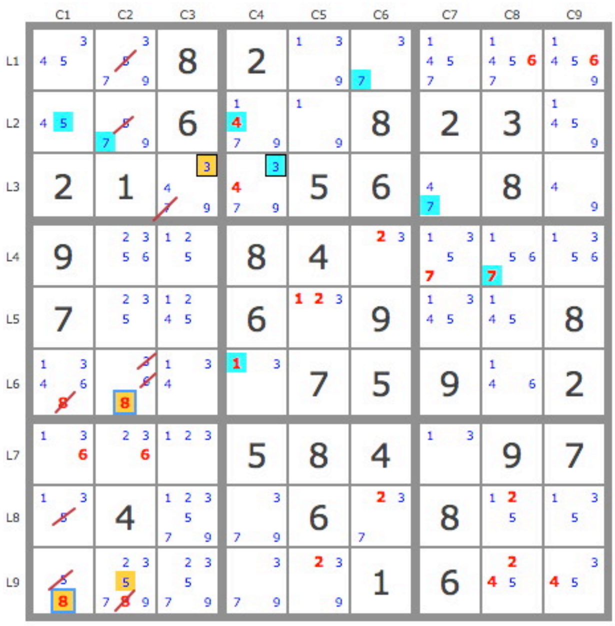
But the subject is theorem 2.

The tracks P(3r3c3) and P(3r3c4) are blocked in development. To unlock I will consider another pair, the one in box r7c6, 4r7c6 and the set 23r7c6. As you say elsewhere, necessarily P(4r7c6) or P(23r7c6) is valid and the other is invalid, because these two tracks are conjugated. So I'm going to build them.
P(4r7c6)={4r7c8,... }
P(23r7c6)=P(2r7r6)∩ P(3r7r6), donc je construis séparément P(2r7r6) et P(3r7r6).
Pour faciliter l'explication, je marque toutes ces pistes avec des couleurs sur le puzzle.
P(3r3c3) en jaune, P(3r3c4) en bleu, P(2r7r6) en vert et P(3r7r6) en violet.



As can be seen, the two tracks P(2r7r6) green and P(3r7r6) purple are opposed to the yellow track P(3r3c3) by the 5. so according to theorem 4 TDP part 3, the candidates of the blue track P(3r3c4) are candidates of the green and purple tracks, therefore are candidates of track P(23r7c6).
Thanks to this I can develop P(23r7c6) which leads to a contradiction (I'll let you check it out).

Finally, 23r7c6 can be eliminated, 4r7c6 is the solution and the blue track can be expanded as shown in the following figure.
Several eliminations are then possible (candidate crossed out in red) and two candidates 8 are solutions.



Opposite tracks also find their interest in grid analysis to choose the extensions of a track, as will be defined in Part 4.

Part 4

Introduction

In this penultimate part I discuss the notion of extending a track.

For the notions mentioned in this part 4 (track, anti-track, conjugated tracks, etc...) I refer the reader to parts 1, 2 and 3.

Extension of a track.

I would like to remind you that the term " track P" refers to either a track or an anti-track.

Definition :
P and Q being any two tracks, a P-track marked P.Q is the set P.Q={Bj, j=1,2, ...} made up of candidates Bj from track P and track Q, as well as all the candidates Bj that would be placed on the grid IF the candidates from track P and track Q were placed.
It is said that the P runway has been extended by the Q runway.

As for anti-tracks and tracks, if the P-track P.Q encounters a contradiction it is invalid, if it is made up only of solution candidates it is valid.

Then let's define the extension of a track P.

Definition :
Q(E) and Q'(E) being the track and anti-track generated by any set E, P being any track, the P-track P=P.Q(E) is an extension of the track P if the P-track P.Q'(E) is invalid.*
Q(Ei) being the tracks generated by the Ei components of E, the P-tracks P.Q(Ei) form the branches of the P extension.

I will not develop too much here this concept (resolution tree) to keep it simple and I will only state, without giving demonstrations, some practical results.

Theorem 1:
Let E1 and E2 be a pair of sets (*) contained in a track P.
If P.Q(E1) is invalid, then P.Q(E2) is an extension of P.

(*) *Two disjoined sets form a pair of sets when their meeting is made up of all the candidates of the same cell or all the candidates of the same occurrence from the same zone (row, column or block).*
A pair of sets contained in a track P is a pair of sets from which the candidates who see P are removed.

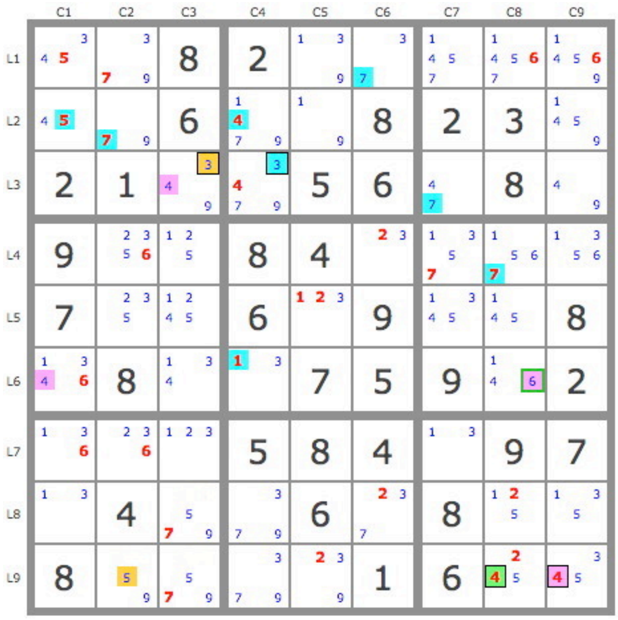
Theorem 2 :
Tracks P1 and P2 being conjugated,
*1) If an extension P*1 of P1 is invalid, then an extension P*2 of P2 is valid.*
*2) If a candidate sees both a candidate of a P*1 extension of P1 and a candidate of a P*2 extension of P2, he can be eliminated from his cell.*
*3) If a candidate is common to a P*1 extension of P1 and a P*2 extension of P2, he is a solution in his cell.*

These two theorems are powerful tools for solving difficult puzzles.

Here is an example of how to use this concept to better understand it.

G = ..82.......6....3.21..56.8.9..84....7..6.9..8....75..2...58..97.4....8.......16..

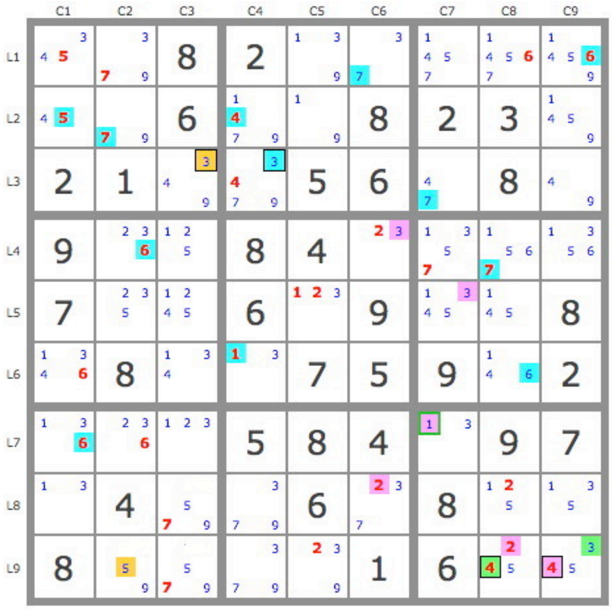
This is the grid studied as an example in the TDP Part 3. We resume its study at the stage where we left it in part 3 with two tracks combined by the 3r3, P(3r3c4) blue and P(3r3c3) yellow.
To further develop the blue runway, we consider its extension P(3r4c3).P(4r9c89) because P(3r4c3).P'(4r9c89) is obviously invalid as having no candidate of occurrence 4 in block B9.
We see on the figure below where we have drawn the two branches P(3r4c3).P(4r9c8) in green and P(3r4c3).P(4r9c9) in purple, that they share the 6r6c8 which is therefore a candidate of P(3r4c3).P(4r9c89).
P(3r4c3).P(4r9c89) =P(3r4c3) U {6r6c8, 6r4c2, 6r1c9, ...}
This eliminates the 5r4c2 that sees the yellow track and the extension of the blue track (theorem 2).



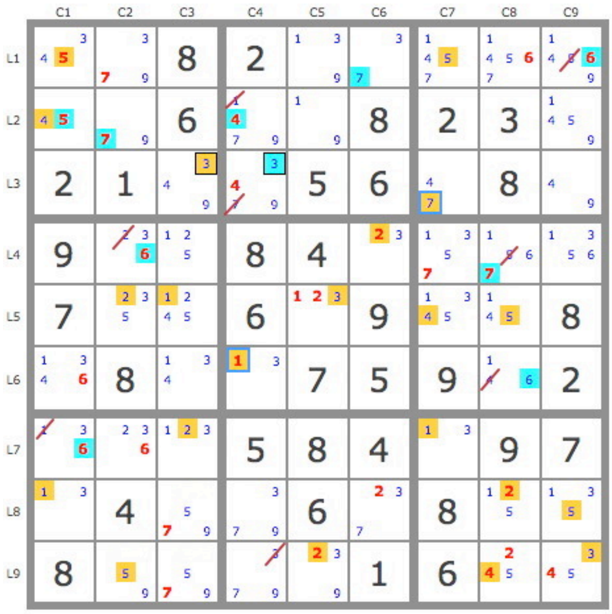
Obviously, it would be tedious to eliminate one or a few candidates, that is not the primary goal. The goal is now to develop the yellow track to increase the number of eliminations.

D'un point de vue pratique on trace avec la même couleur les candidats de l'extension d'une piste P dès qu'ils sont identifés.

Maintenant, on considère l'extension P(3r3c3).P(r9c89) et on voit sur la figure suivante que ses branches verte et violette ont en commun le 1r7c7 qui est donc un candidat de l'extension, c'est à dire un candidat jaune. Cela permet d'éliminer le 1r7c1.



But if we look closely, we see that the purple branch is invalid with a contradiction in r5c2, which makes the green branch an extension of the yellow track (Théorème 1), and the situation of the grid is the following with a yellow track that develops a lot and allows eliminations and validations :



I'll let you finish the puzzle now, and I'll give other examples of how to solve it with the TDP.

I will give in the forum (here) examples of TDP resolution for puzzles of different levels of difficulty.

Part 5

Introduction

I will conclude this presentation of the TDP with a paragraph dedicated to the uniqueness of the solution in puzzles, and therefore to multiple solution puzzles.

Contrary to the customs of the sudoku community, I consider that multiple solution puzzles are of interest (to me) from a theoretical point of view. So I was interested in it in order to establish track properties that apply to all classic 9x9 puzzles.

Scope of the TDP

The definitions, properties and theorems given in Parts 1, 2 and 4 are applicable to all classic 9x9 puzzles, whether they are single or multiple solutions.

The same is not true for opposing tracks (part 3) of multiple solution puzzles for which, it is obvious, ownership can no longer be stated (theorem 1). Two opposing paths may be valid and lead to two different solutions.

Also, if we are not sure that a puzzle is a single-solution puzzle and if we misuse the opposite paths we will be led to fewer solutions than the number of solutions in the puzzle.

On the other hand, theorem 2 of TDP part 3 is valid but its demonstration must be adapted.

Finally, the theorem 3 of TDP part 3 must be expressed differently, like this:

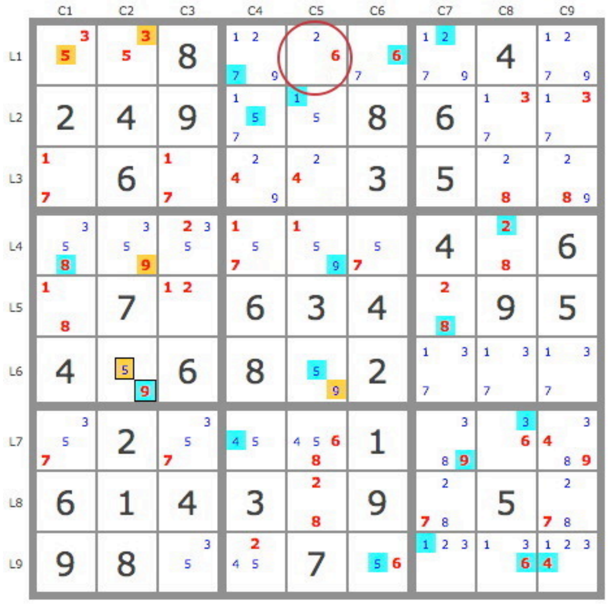
Si deux pistes conjuguées Q1 et Q2 sont opposées à la même piste P1, toute piste conjuguée P2 de la piste P1 est composée de candidats communs à toutes les solutions S1, S2, S3, ...du puzzle, c'est-à-dire P2 $\subseteq \cap$*Si*

Uniqueness of the solution

Whenever we deal with a puzzle that we do not know has a single solution, the notion of the opposite track must be considered with caution, at the risk of not seeing that the puzzle has several solutions.

The same applies to methods that assume that the puzzle is a single-solution puzzle such as the single rectangle (UR), BUGGs, etc. that I call here "uncertain configurations", for two reasons:
- We cannot prove that a puzzle has a unique solution by using a principle that already sets the puzzle as having a unique solution. It's obvious!
- If we use an uncertain configuration in the resolution, we will find some solutions to the puzzle, but maybe not all of them.

On the puzzle below for example, we consider the two conjugated tracks P(9r6c2) marked in blue and P(5r6c2) marked in yellow.
If we treat the puzzle as having a single solution and therefore consider 8r4c1 as a candidate for the blue track in order to avoid UR 35r1c23-35r4c23, we end up with a contradiction in the cell r1c5.
The blue track is therefore invalid and the yellow track is valid, i.e. 9r6c2 can be eliminated and all candidates marked in yellow are solutions.
This result is not false, but by doing this we do not see at this stage of resolution that the puzzle actually has several solutions, and that the 9r6c2 is part of two possible solutions S1 = 9r6c2 -> 5r1c2 -> 3r1c1 -> etc... and S2 = 9r6c2 -> 5r4c2 -> 3r1c2 -> etc...



That said, you can exploit an uncertain configuration in any puzzle as long as you use it to make an extension of a track (see definition of an extension in TDP part 4).
Thus, in this previous puzzle we can develop P(9r6c2) by the two branches of an extension P(9r6c2).P(8r4c1) and P(9r6c2).P'(8r4c1).
P(9r6c2).P(8r4c1) being invalid (as we have just seen), the extension of P(9r6c2) is done with P(9r6c2).P'(8r4c1) which leads to the two solutions S1 and S2.

This approach has the advantage of dealing with a puzzle without worrying about whether or not it has a single solution, and if it has a single solution, the extension branch obtained with the UR will lead to a contradiction.

Difficulty level of a puzzle

This approach to uniqueness has led me to establish a level of difficulty for single-solution puzzles, called the TDP level, that is totally different from those commonly used.

Definition :
The size of a puzzle resolution by the TDP is equal to the number of invalid conjugated tracks used in practice to solve the puzzle.

For example, if a first set of conjugated tracks that reduces the puzzle was used to solve a puzzle, then a second set of conjugated tracks that completes the puzzle, the resolution size is 2.

Another example is a very difficult puzzle that admits a backdoor P(E).
With T&E you can find this backdoor, but this does not mean that the resolution size is 1, its level of difficulty must be established by showing that the anti-backdoor P'(E) is invalid.

Definition :
The TDP level of a puzzle is equal to the smallest possible resolution size.

In the above example, we cannot say that the TDP level is 2, but simply that it is less than or equal to 2.

For example, it can be established that Easter Monster has a TDP level ≤ 13, AI-Escargot has a TDP level ≤ 10.

Robert Mauriès
November 16, 2019